Finding the optimal location for company headquarters and warehouse

Key Stage: 3

Strand: Measures, Shape and Space

Learning Unit: Rectangular coordinate system
Centres of triangles
Inquiry and investigation

Objectives:
(i) To enrich students' experience in applying the mid-point formula, distance formula, and centres of triangles in solving real-life problems
(ii) To enhance students' abilities in making assumptions and applying mathematical concepts in modelling
(iii) To create a virtual model using mathematics software

Prerequisite Knowledge:
(i) Understanding the mid-point formula
(ii) Understanding the Pythagoras' theorem and distance formula
(iii) Basic understanding of the centres of triangles

Resources Required: Desktop or tablet computers with GeoGebra software or Internet connection

Background Information:
Finding an optimal location for a building (e.g., headquarters, warehouse, library and hospital) is an important issue in industrial engineering and city planning. This task involves careful consideration of various factors to ensure efficient operations and maximum accessibility for the intended purpose of the facility. The main goal of the following modelling activities is to use mathematics to find the optimal locations for the headquarters and warehouse of a company. This helps the company communicate with their stores and deliver their products efficiently.

By delving into the complexities of finding the optimal locations for the headquarters and warehouse of a company, this set of activities conveys the essence of prescriptive modelling. Specifically, students are required to find the "best" solution for a given problem. The modelling outcomes achieved through these activities facilitate meaningful discussions and reflections among students. As they navigate through scenarios involving equidistance and the shortest distance, students are prompted to apply the mid-point formula, distance formula, and centres of triangles to derive solutions.

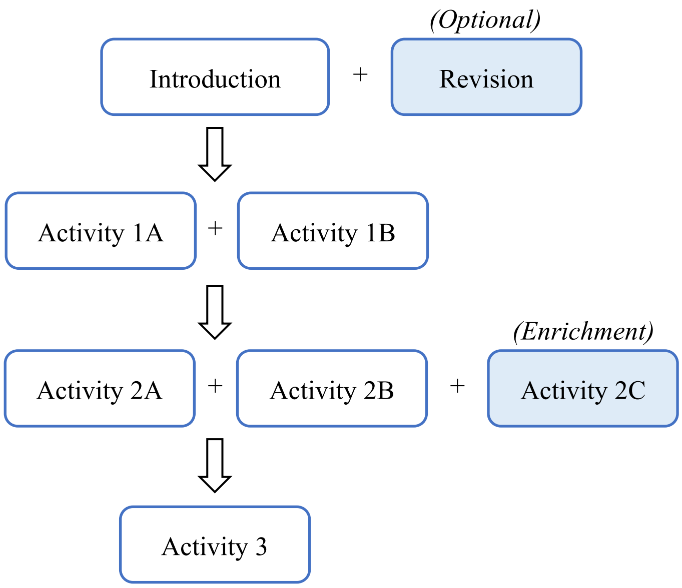
Beyond mathematical concepts, this exploration nurtures students' modelling competence to make informed assumptions and recognise limitations of a model. The exploration extends further with the activity, where students leverage information technology to create a virtual model of the optimal location of the headquarters and warehouse. This integration of mathematics and technology not only enriches students' learning experience but also demonstrates the relevance of mathematical modelling in our increasingly digital world.

Description of the Activities:

There are three main activities in this resource package:

- Activity 1: To find locations for a headquarters (1A) and warehouse (1B) considering two stores.
- Activity 2: To find locations for a headquarters (2A) and warehouse (2B and 2C) considering three stores.
- Activity 3: To create virtual models for finding the locations.

Based on student abilities and school contexts, teachers can consider adopting the following approach to activity plan.



Based on Yong et al.'s (2015) framework of the mathematical modelling process, the following table summarises the elements that teachers can discuss with students in the corresponding questions.

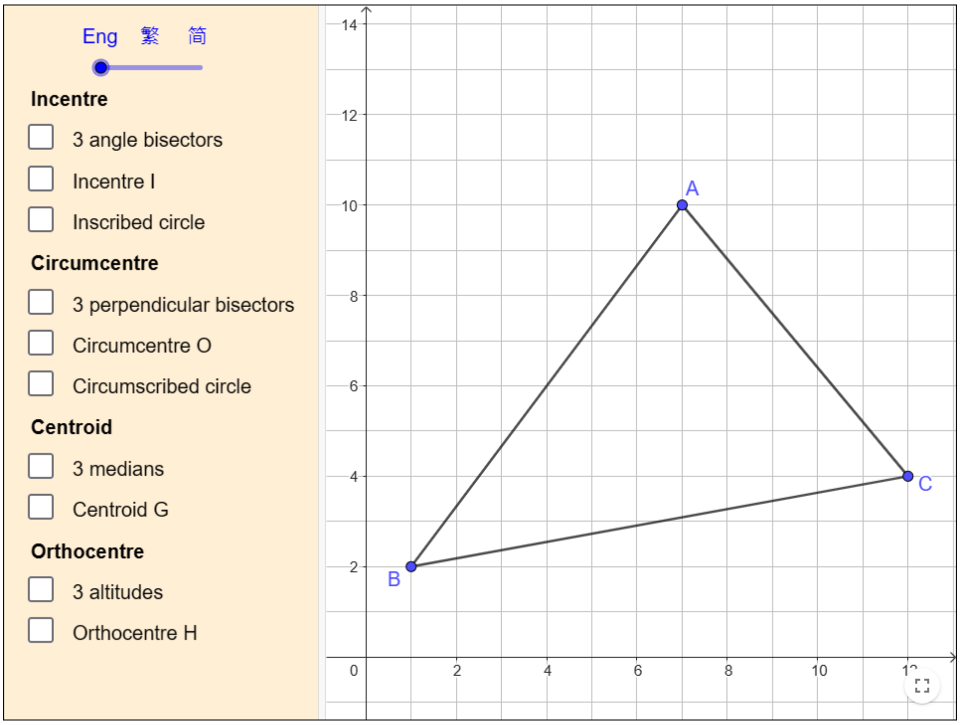
Phase	Elements	WS1A	WS1B	WS2A	WS2B	WS2C	WS3
Define	Define questions of interest	Cover page, 1	7	1	5	7	
	Identify variables and parameters	6					
Translate	Identify governing principles	1	7	1	5		
	Make simplifying assumptions	5	9	3			
	Formulate mathematical model	3					2–4
Analyse	Select appropriate math tools & Solve mathematical problem	2, 4	7	2	6	7	1–4
	Determine or estimate parameters						
	Validate solution			2	6		
Interpret	Visualise solution	2, 4	7	2	6	7	2–4
	Draw appropriate conclusions & Communicate results		8	2, 4		7	

<u>Revision (based on students' ability)</u>

The part is to recall students' prerequisite knowledge of the centres of triangles.

Pedagogical recommendations:

1. In Questions 1 to 4, the teacher can use the following applet to visualise the special lines of a triangle and the centres. The coordinates of A, B and C can be changed, thereby facilitating students' observation on the positions of the centres.



Link: https://www.geogebra.org/m/btwzyahk

<u>Suggested solution:</u>

1. Incentre I is the point of intersection of the three <u>angle bisectors</u> in a triangle.
2. Circumcentre O is the point of intersection of the three <u>perpendicular bisectors</u> in a triangle.
3. Centroid G is the point of intersection of the three <u>medians</u> in a triangle.
4. Orthocentre H is the point of intersection of the three <u>altitudes</u> in a triangle.

<u>Activity 1A (refer to Worksheet 1)</u>

In this activity, the problem focuses on how to find a location for a headquarters where its distances from two different stores are the same and the shortest. The activity allows students to apply the concept of the mid-point in the real-world problem.

Pedagogical recommendations:

1. In Question 1, the teacher enhances students' ability to represent a real-world problem using mathematical terms. The teacher can facilitate the discussion by asking students to draw two arbitrary points representing stores *A* and *B*. Notice that students may describe the requirement as "$QA = QB$." However, this does not sufficiently fulfil the requirement because *A*, *Q* and *B* may not be colinear even though $QA = QB$. For more capable students, the teacher may consider discussing the graphs in 2003 HKCEE Mathematics Paper 2 Question 31 and 2016 HKDSE Mathematics (Compulsory Part) Paper 2 Question 26.

   <u>Suggested solution:</u>

   Q is the mid-point of *A* and *B*.

2. With the experience gained from Question 1, students are tasked to locate the headquarters in some specific problems.

   <u>Suggested solution:</u>

   (a) $Q = (3, 2)$

   (b) $Q = (4, 2)$

   (c) $Q = (3, 2)$

3. The teacher can introduce students to the concept of a model. In this scenario, the model can be used to find the location of the headquarters *Q*. This task requires their knowledge of the mid-point formula.

   <u>Suggested solution:</u>

   $$Q = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$$

4. Question 4(a) allows students to apply their formulated model to find the location of the headquarters *Q*. In Question 4(b), the distance formula is used to calculate the distance from *Q* to each of the stores *A* and *B*.

<u>Suggested solution:</u>

(a) $A = (2, 6)$ and $B = (10, 2)$

$$Q = (\frac{2+10}{2}, \frac{6+2}{2})$$
$$= (6,\ 4)$$

(b) $$QA = \sqrt{(2-6)^2 + (6-4)^2}$$
$$= \sqrt{20}$$
$$= 4.47 \text{ km}$$

$$QB = QA$$
$$= 4.47 \text{ km}$$

5. Toward the end of Activity 1A, the teacher can facilitate students' discussion on the assumptions of the formulated model. This can enhance students' abilities in making assumptions in modelling. The following are some possible discussion outcomes.
   - 2D-model: Assuming the ground is flat. In reality, the surface of the Earth is not flat. If the distances between the headquarters and the two stores are very long, we have to consider the curvature of the Earth's surface.
   - Unobstructed transmission: There are no obstacles (e.g., hills or buildings) which block the transmission of the radio wave between the headquarters and the two stores.

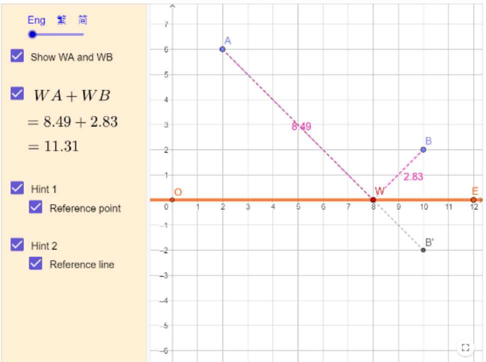
6. The teacher can further encourage students to discuss other factors that should be considered when finding the location. This can enhance their awareness and critical thinking regarding the feasibility of approaches. The following are some possible discussion outcomes.
   - Feasibility of construction: The construction of the headquarters at the selected location involves assessing whether the location meets the necessary building regulations and environmental considerations.
   - Construction costs: The expenses associated with building the headquarters at the selected location include the costs of land acquisition, construction materials, and labour, etc.

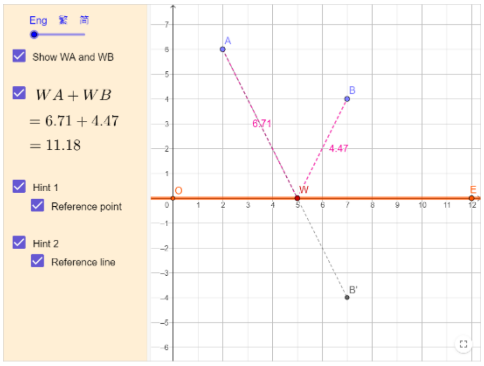
Activity 1B (refer to Worksheet 1)

In this activity, the problem focuses on how to find a location for a warehouse where its distances from two different stores are the shortest. The activity allows students to apply the knowledge of transformations of a point in the real-world problem.

Pedagogical recommendations:

7. The teacher can facilitate students' mathematical investigation by using the following applet. The coordinates of *A* and *B* can be changed, thereby facilitating their observation on different scenarios, such as the following.





Link: https://www.geogebra.org/m/a6b9jb2g

Suggested solution:

Based on the requirements, *WA* + *WB* should be the smallest (or shortest).

8. As finding the coordinates of *W* involves senior secondary mathematics (equations of straight lines), Question 8 only requires students to describe the ways of finding its location.

Suggested solution:

First, Point *B* is reflected about *OE* and the image is *B'*.

Second, we draw a straight line *AB'*.

Then, the point of intersection of *OE* and *AB'* is the location of the warehouse *W*.

9. Similar to Question 6, the teacher can encourage students to consider any possible constraints when searching for the location of the warehouse. The following are some possible discussion outcomes.
   - Possible constraints: The construction costs of building the warehouse beside a distributor road may be very high. In terms of city planning, the area close to a distributor road may be allocated for commercial or residential purposes, rather than industrial purposes.
   - To remedy the constraints, we can first identify the optimal location of *W*. Then, we can search for other feasible locations as close to the optimal location as possible.

Activity 2A (refer to Worksheet 2)

Similar to Activity 1A which involves two stores, the problem in this activity focuses on how to find a location for a headquarters where its distances from three different stores are the same. The activity allows students to delve into the property of the circumcentre of a triangle in the real-world problem, as compared with other three centres of a triangle.

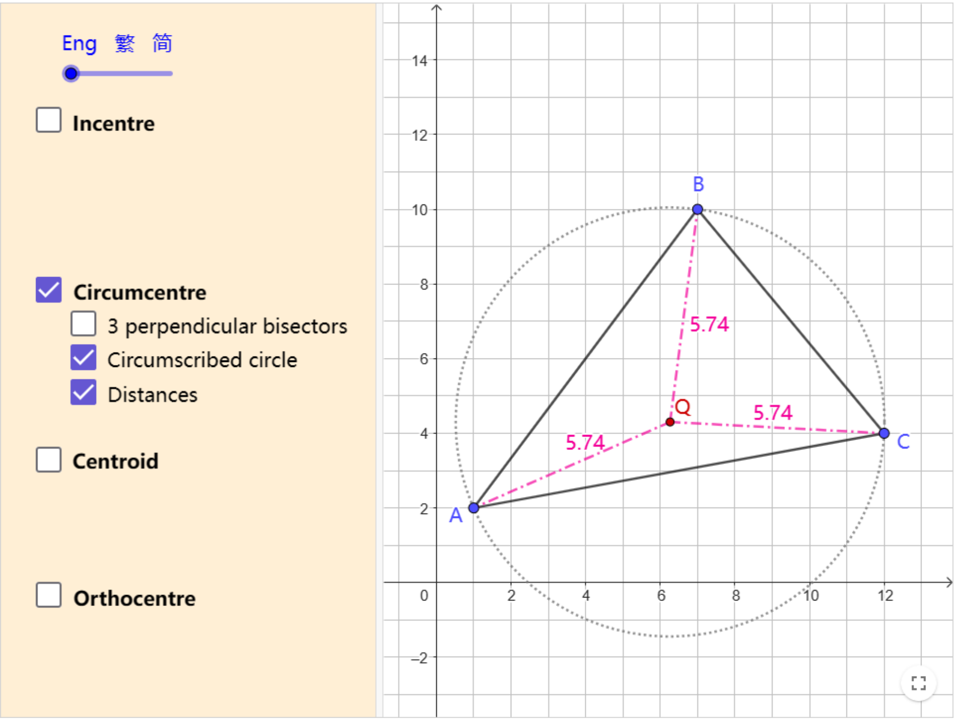
Pedagogical recommendations:

1. In Question 1, the teacher enhances students' ability to represent a real-world problem using mathematical terms.

   Suggested solution:

   $QA = QB = QC$ (or QA, QB and QC are equal.)

2. Based on students' prerequisite knowledge regarding the centres of triangles, they are tasked to select an appropriate model to solve the problem. The following applet can be used to facilitate their mathematical investigation.



   Link: https://www.geogebra.org/m/gm6ayhap

Suggested solution:

Your choice ☑	Centres	Distance (correct to the nearest 0.01 km)		
		QA	***QB***	***QC***
	Incentre	7.18	4.23	5.20
✓	Circumcentre	5.74	5.74	5.74
	Centroid	6.57	4.68	5.50
	Orthocentre	8.43	2.65	5.66

3. With the experiences in Activity 1A, students should be more able to identify the assumptions made in modelling. The following are some possible discussion outcomes which are similar to those in Question 5 in Activity 1A.
   - 2D-model: Assuming the ground is flat. In reality, the surface of the Earth is not flat. If the distances between the headquarters and the three stores are very long, we have to consider the curvature of the Earth's surface.
   - Unobstructed transmission: There are no obstacles (e.g., hills or buildings) which block the transmission of the radio wave between the headquarters and the three stores.

4. Then, the teacher can prompt students to consider the limitations of the model. Specifically, the model cannot minimise the total distance from the headquarters Q to the three stores A, B and C.

   Suggested solution:

Centres	Distance (correct to the nearest 0.01 km)			
	QA	***QB***	***QC***	**Total**
Incentre	7.18	4.23	5.20	16.61
Circumcentre	5.74	5.74	5.74	17.22
Centroid	6.57	4.68	5.50	16.75
Orthocentre	8.43	2.65	5.66	16.74

The selected location (the circumcentre of $\triangle ABC$) cannot minimise the total distance from Q to A, B and C. As shown in the above Table, the distance (circumcentre: 17.22 km) is greater than the three other locations, including the incentre (16.61 km), centroid (16.75 km) and orthocentre (16.74 km) of $\triangle ABC$.

<u>Activity 2B (refer to Worksheet 2)</u>

In this activity, the problem focuses on how to find a location for a warehouse where its distances from three sides of the triangle are equal. The activity allows students to delve into the property of the incentre of a triangle in the real-world problem, as compared with other three centres of a triangle.

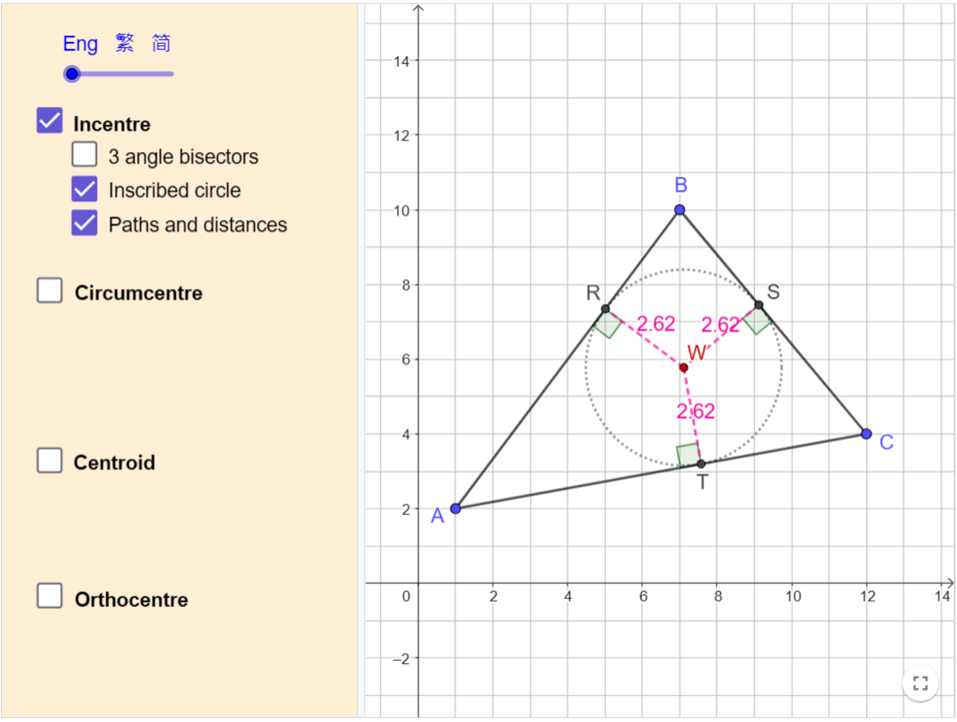
Pedagogical recommendations:

5. In Question 5, the teacher enhances students' ability to represent a real-world problem using mathematical terms.

   <u>Suggested solution:</u>

   $WT \perp AC$ (or WT is perpendicular to AC.)

6. Based on students' prerequisite knowledge regarding the centres of triangles, they are tasked to select an appropriate model to solve the problem. The following applet can be used to facilitate their mathematical investigation.



   Link: https://www.geogebra.org/m/bgpwnvje

Suggested solution:

Your choice ☑	**Centres**	**Distance (correct to the nearest 0.01 km)**		
		WR	***WS***	***WT***
✓	Incentre	2.62	2.62	2.62
	Circumcentre	2.83	4.21	1.32
	Centroid	2.53	3.24	2.27
	Orthocentre	1.94	1.30	4.15

Activity 2C (refer to Worksheet 2)

This enrichment activity involves novel mathematical investigation and is suitable for more capable students. In this activity, students explore how to locate the warehouse W on a side of $\triangle ABC$. The location should minimise the total distance of WA, WB and WC.

Pedagogical recommendations:

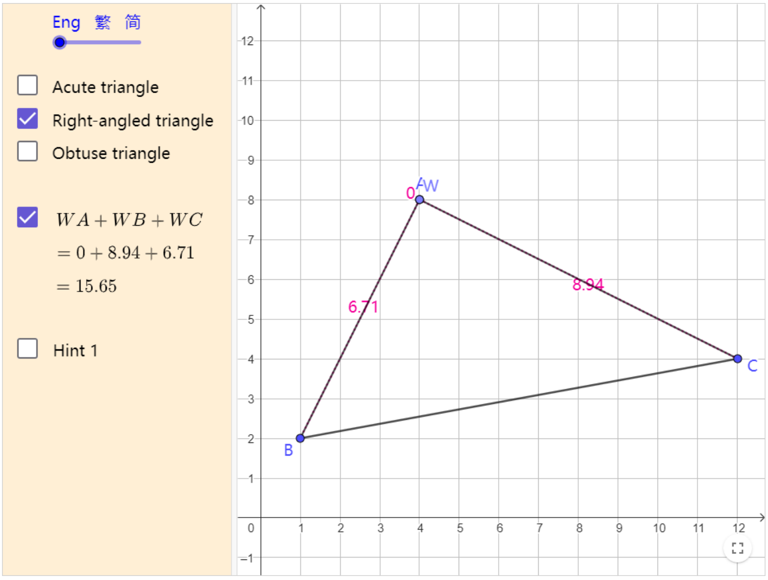
7. Question 7 involves a consideration of different cases of $\triangle ABC$. The teacher may ask students to consider the case of a right-angled triangle first, followed by an obtuse triangle, and finally an acute triangle. The first two cases (i.e., right-angled triangle and obtuse triangle) may be easier because the warehouse is located at a vertex of the triangle. The third case (i.e., acute triangle) may be more challenging and require more guidance from the teacher.

   Students can use the provided applet to facilitate their mathematical investigation: https://www.geogebra.org/m/gxtnz4cu

   Suggested solution:

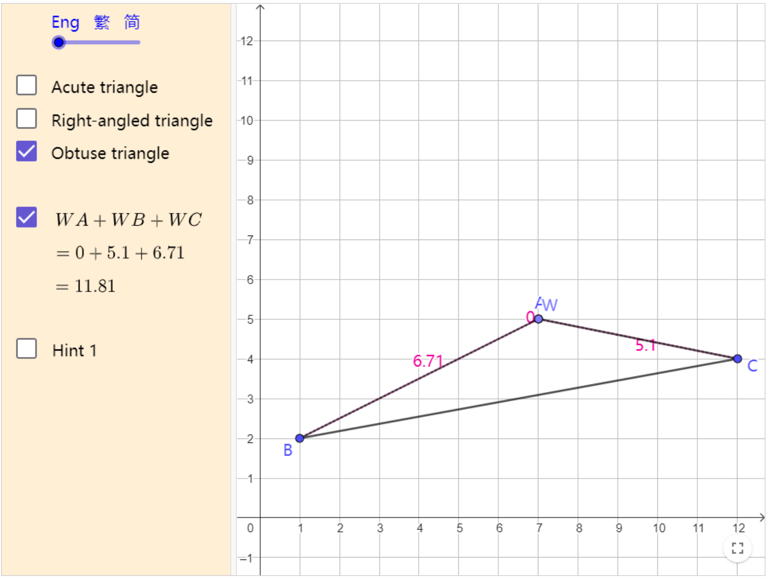
   Case I: Right-angled triangle

   W is located at the vertex formed by the two shorter sides of the triangle.



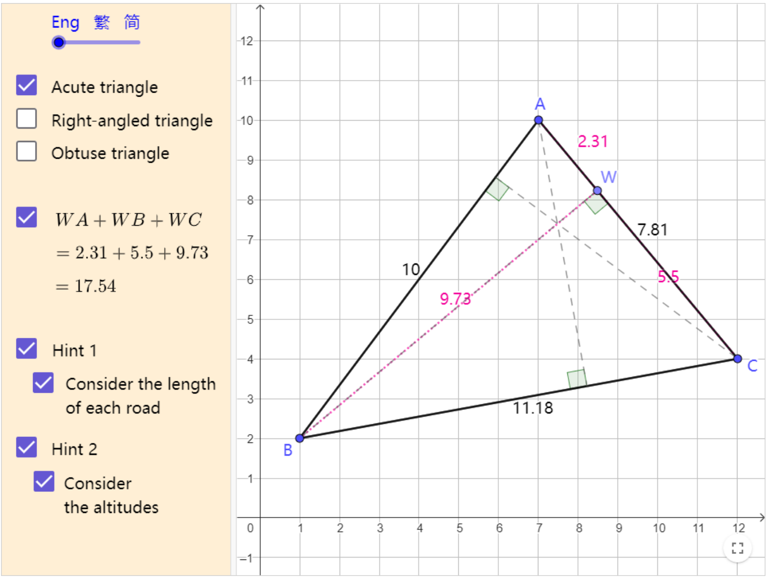
Case II: Obtuse triangle

W is located at the vertex formed by the two shorter sides of the triangle.



Case III: Acute triangle

W is located at the foot of a perpendicular on the shortest side (from its corresponding vertex) of the triangle.



<u>Activity 3 (refer to Worksheet 3)</u>

In this activity, students will create a virtual model formulated in Activities 1A (the mid-point of two points), 2A (the circumcentre of a triangle) and 2B (the incentre of a triangle).

Pedagogical recommendations:

1. Students are introduced to the use of GeoGebra in creating virtual models. https://www.geogebra.org/classic is the online application of GeoGebra. Alternatively, the teacher and students can install GeoGebra on computers.
   Please visit: https://www.geogebra.org/download

2. Following the step-by-step instructions, students create a virtual model to locate the headquarters Q which is the mid-point of stores A and B in Activity 1A. During the construction, the teacher can recap the concept of the mid-point and the assumptions involved in modelling.

3. Following the step-by-step instructions, students create a virtual model to locate the headquarters Q which is the circumcentre of stores A, B and C in Activity 2A. During the construction, the teacher can recap the property of the circumcentre of a triangle and the assumptions involved in modelling.

4. Following the step-by-step instructions, students create a virtual model to locate the warehouse W which is the incentre of stores A, B and C in Activity 2B. During the construction, the teacher can recap the property of the incentre of a triangle and the assumptions involved in modelling.

Reference:

Yong, D., Levy, R., & Lape, N. (2015). Why no difference? A controlled flipped classroom study for an introductory differential equations course. *PRIMUS, 25*(9–10), 907–921.

Suggested lesson plans and teaching flow

Teaching time: 70 minutes or a double lesson

Time (mins)	Objectives	Teaching activities and processes	Resources/ remarks
10	• To arouse students' interest • To recall prerequisite knowledge	1. The teacher arouses students' interest by discussing the real-world scenario. 2. The teacher recalls students' prerequisite knowledge of the centres of triangles.	WS cover page WS Revision Q1–4
15	• To apply the concept of the mid-point • To introduce the concept of a model • To apply the formulated model	1. The teacher enhances students' ability to represent a real-world problem using mathematical terms. 2. The teacher facilitates the discussion by asking students to draw two arbitrary points representing stores *A* and *B*. 3. Students are tasked to locate the headquarters in some specific problems. 4. The teacher introduces students to the concept of a model. 5. Students apply their knowledge of the mid-point formula to formulate the model. 6. Students apply their formulated model to find the location of the headquarters *Q*. 7. Students use the distance formula to calculate the distance from *Q* to each of the stores *A* and *B*.	WS1A Q1–2 WS1A Q3 WS1A Q4

Time (mins)	Objectives	Teaching activities and processes	Resources/ remarks
	• To enhance abilities in making assumptions and identifying other factors	8. The teacher facilitates students' discussion on the assumptions of the formulated model. 9. The teacher encourages students to discuss other factors that should be considered when finding the location.	WS1A Q5–6
10	• To apply the knowledge of transformations of a point	1. The teacher facilitates students' mathematical investigation by using the provided applet. 2. Students describe the ways of finding the location for the warehouse.	WS1B Q7–8
	• To enhance abilities in identifying possible constraints	3. The teacher encourages students to consider any possible constraints when searching for the location of the warehouse.	WS1B Q9
10	• To delve into the property of the circumcentre of a triangle	1. The teacher enhances students' ability to represent a real-world problem using mathematical terms. 2. Students are tasked to select an appropriate model (from the four centres of triangles) to solve the problem. 3. Students use the provided applet to facilitate their mathematical investigation.	WS2A Q1–2
	• To enhance abilities in making assumptions and identifying limitations in modelling	4. The teacher facilitates students' discussion on the assumptions of the formulated model. 5. The teacher prompts students to consider the limitations of the model.	WS2A Q3–4

Time (mins)	**Objectives**	**Teaching activities and processes**	**Resources/ remarks**
10	• To delve into the property of the incentre of a triangle	1. The teacher enhances students' ability to represent a real-world problem using mathematical terms. 2. Students are tasked to select an appropriate model (from the four centres of triangles) to solve the problem.	WS2B Q5–6
	• To identify the best solution in different cases	3. The teacher facilitates students' mathematical investigation by using the provided applet. 4. Students describe the ways of finding the location for the warehouse in different cases.	WS2C Q7
15	• To create the virtual models formulated in this activity • To conclude the activity	1. Students create a virtual model (the mid-point of two points) in Activity 1A. 2. The teacher recaps the concept of the mid-point and the assumptions involved in modelling. 3. Students create a virtual model (the circumcentre of a triangle) in Activity 2A. 4. The teacher recaps the property of the circumcentre of a triangle and the assumptions involved in modelling. 5. Students create a virtual model (the incentre of a triangle) in Activity 2B. 6. The teacher recaps the property of the incentre of a triangle and the assumptions involved in modelling.	WS3 Q1–4